PRACTICAL 1.11

STUDY OF FORCED OSCILLATIONS

Objective: an experimental study of motion of an object in a non-inertial frame of reference; study of the amplitude and phase characteristics of forced oscillations.

Tools and equipment: experimental setup, stopwatch, magnifying glass.

INTRODUCTION

In an inertial frame of reference forced oscillations of a compound pendulum in a gravitational field are described by the equation

$$I_z \frac{d^2\alpha}{dt^2} = M_g + M_f + M_d \ , \qquad (1)$$

where α is the angle of displacement of the pendulum from the equilibrium position; M_g, M_f and M_d are the moments of gravitational force, friction force and the driving force impacting on the pendulum, respectively; I_z is the moment of inertia of the pendulum with respect to the axis of rotation.

For small values of the angle of displacement $M_g = -mgl \sin\alpha \approx -mgl\alpha = -k\alpha$, and the friction torque can be considered proportional to the angular velocity of the pendulum. If the external (driving) force changes harmonically over time, equation (1) takes the form

$$I_z \frac{d^2\alpha}{dt^2} = -k\alpha - r\frac{d\alpha}{dt} + M_0 \cos\omega t \quad \text{or} \quad \frac{d^2\alpha}{dt^2} + 2\beta\frac{d\alpha}{dt} + \omega_0^2\alpha = \frac{M_0}{I_z}\cos\omega t , \qquad (2)$$

where $k = mgl$ is the coefficient of the moment of gravitational force bringing the pendulum back to the equilibrium state; m, l and g are the mass, distance from the pendulum's center of mass to the axis of suspension and acceleration of the free-fall; r is the coefficient of friction; $\beta = r/(2I_x)$ is the damping coefficient; $\omega_0 = \sqrt{\frac{k}{I_z}}$ is the angular frequency of intrinsic oscillations of the pendulum, ω is the angular frequency of torque of the driving force, M_0 is the amplitude of torque of the driving force.

The motion of the pendulum described by this equation is complicated. It is basically a sum of the forced harmonic oscillation with a certain angular amplitude α_0 and the angular frequency ω, and free oscillations with a frequency $\omega_{fr} = \sqrt{\omega_0^2 - \beta^2}$ decaying over time. However, after the transient process, when the free oscillation is damped, the pendulum starts to oscillate harmonically (stationary mode) with a certain phase shift φ with respect to the driving force

$$\alpha = \alpha_0 \cos(\omega t - \varphi), \qquad (3)$$

where

$$\alpha_0 = \frac{M_0}{I_z\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}} \quad \text{and} \quad tg\varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \ . \qquad (4)$$

EXPERIMENTAL SETUP

In this work, forced oscillations of a compound pendulum with an adjustable point of suspension are studied. The experimental setup consists of two pendulums. Furthermore, one pendulum (the small one) is attached to a structure of the second one (the large one). Fig. 11.1 provides a sketch including both pendulums and their relative positions at some particular time point. The large pendulum presented by a rod *AB* (along which massive loads *D* can be moved) may perform an oscillatory motion in a vertical plane. Relocation of the loads along the rod allows to tune the oscillation period of the large pendulum. The angle of displacement from the vertical γ is measured by a scale *M*. The angular amplitude of the steady-state forced oscillation of the pendulum is determined by a small scale *N*.

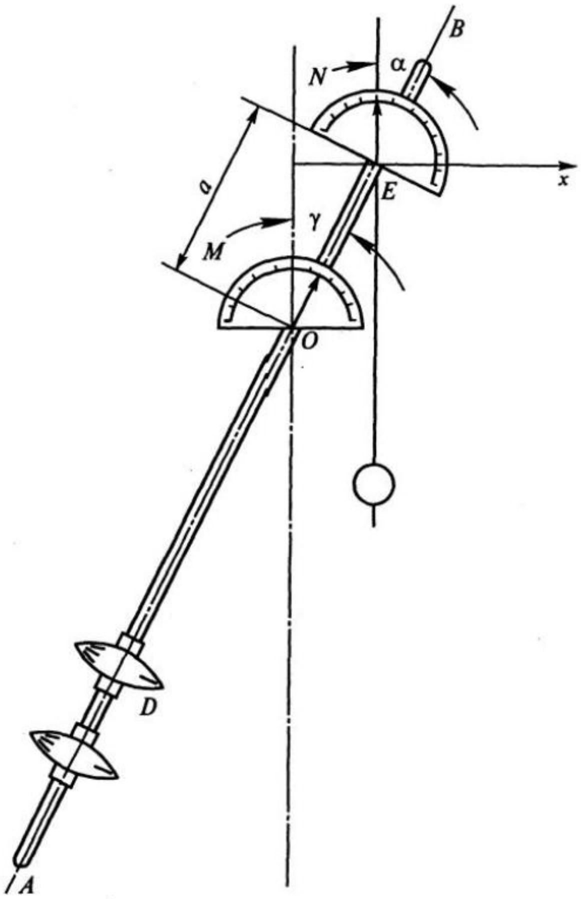

Fig. 11 1

As one can see from the sketch, the small pendulum is involved into two independent motions: oscillatory and translational. The latter is performed together with the point of suspension (for small oscillations of the large pendulum). Thus, only the oscillatory motion can be further considered, once switched to the frame of reference related to the point of suspension.

Now, let us write down the equations of motion of the small pendulum in this frame of reference. For small oscillations of the large pendulum it can be assumed that the point of suspension oscillates along a straight line *X* according to the law $x = X_0 \cos(wt)$, where X_0 is the amplitude of displacement of the point of suspension, ω is the angular frequency of oscillations of the large pendulum. Consequently, the selected frame of reference is non-inertial, since it moves with an acceleration $a_{\text{frame}} = -X_0\omega^2\cos(\omega t)$. To write down the equation of motion of the small pendulum in such a frame of reference one needs to take into account the force of inertia $F = -m\frac{d^2x}{dt^2} = mX_0\omega^2\cos\omega t$ (in addition to the gravitational and friction forces). This is the driving force for oscillations of the small pendulum, and its maximum torque is reached at the center of mass of the small pendulum — $M_0 = mX_0 l\omega^2$, where *l* is the distance from the point of suspension of the small pendulum to its center of mass.

Thus, motion of the small pendulum is defined by impact of three forces: the gravitational force, the friction force (proportional to velocity) and the periodic (sinusoidal) driving force. As an outcome, the pendulum starts to swing, i.e. it possesses an angular acceleration. Therefore, introducing the same notation as in (2), the equation of the oscillatory motion of the small pendulum takes the form

$$\frac{d^2\alpha}{dt^2} + 2\beta\frac{d\alpha}{dt} + \omega_0^2\alpha = \frac{M_0}{I_z}\cos\omega t\ .$$

The amplitude of these oscillations and the phase difference between the driving force and the displacement are determined by the equations (3) and (4).

Design of the experimental setup makes it possible to vary frequency of the driving force and, therefore, allows to get the experimental dependence $\alpha_0=\alpha(\omega)$ (the amplitude characteristics of the pendulum).

The phase characteristics of oscillations of the small pendulum ($\varphi = \varphi(\omega)$) can be calculated from the formula (4), and the damping coefficient β and the frequency of the intrinsic oscillations ω_0 are to be determined experimentally.

MEASUREMENT AND DATA PROCESSING

T a s k 1. **Evaluation of frequency and damping coefficient of intrinsic oscillations of the small pendulum**

Value of the damping coefficient is determined from the plot of dependence of amplitude of oscillation of the small pendulum on time. Analytically, this dependence is expressed by the formula $\alpha_m(t)=\alpha_{m0}e^{-\beta t}$. During the relaxation time $\tau = 1/\beta$ the amplitude decreases by a factor e ($e \cong 2{,}7$), i.e. $\alpha_{m\tau} = \alpha_{m\tau}/ e$. Thus, one can easy calculate β, once τ is known.

To determine β of the small pendulum the large pendulum is set to the off position (planes of the knife-edges are leaned against the supporting frame). The small pendulum is provided with an initial displacement of slightly more than 15 divisions of the scale. The stopwatch is activated, when amplitude of the oscillation α_{m0} becomes equal to 15 divisions of the scale, and the time interval corresponding to the amplitude value of $\alpha_m = 13$ divisions of the scale is measured. Then, under the same initial conditions the time intervals, during which the amplitude is reduced to 11, 9, 7, 5, 3 divisions of the scale, are measured. The data obtained is plotted in the coordinates (α_m, t), and all the necessary calculations are carried out.

To determine the angular frequency of free oscillations of a pendulum ω_1 the time of ten complete oscillations of the pendulum t is measured. The frequency is calculated by the formula $\omega_1 = \dfrac{10\cdot 2\pi}{t}$.

Conduct all the measurements required, plot out the graph of dependence $\alpha_m = f(t)$, and determine the frequency of free oscillations and the damping coefficient β of the small pendulum. Write down results of measurements of α_m, t, τ and calculations of β, ω_1, and fill in a table.

Task 2. **Study of the amplitude characteristics of oscillation of the small pendulum $\alpha_m = \alpha(\omega)$**

The large pendulum is set to the on position via the 90° rotation around an axis extending along the pendulum's rod. Please, make sure that it is installed in the correct plane, and the knife-edge is at a proper position in the groove of the supporting frame. The loads are preinstalled at the lowest position. The large pendulum is deflected aside by 6° (the initial deflection is maintained for all measurements), and the period of oscillation of the large pendulum is measured (corresponding to the time of 10 complete oscillations). When the forced oscillations are established (it takes approximately 1.5 minutes, this time should be maintained for all measurement), the amplitude of forced oscillations of the small pendulum is measured. Then the pendulum is set to the off position, and both loads are shifted up by 2 divisions. Bringing the pendulum to the on position, the measurements are repeated, until the upper load reaches the highest position. The data obtained is presented in the form of a graph in the coordinates (α_m, ω).

Conduct all the measurements required, and plot out the graph of dependence of the forced oscillations' amplitude on frequency of the driving force. Write down results of measurements of t, α_m and calculations of ω, and fill in a table.

Task 3. **Study of the phase characteristics of forced oscillations of the small pendulum**

First of all, make sure that at low frequencies, displacement of the small pendulum is practically in phase with displacement of the point of suspension (in antiphase with the lower part of the large pendulum). The phase shift in this case is close to zero.

At high frequencies, displacement of the small pendulum is substantially in antiphase with displacement of the suspension point (in phase with displacement of the lower part of the large pendulum). In this case, the phase shift is close to $-\pi$. The intermediate values are calculated by the formula (4). Perform the calculations for seven values of the frequency: the resonance value and three values corresponding to points on the ascending and descending branches of the amplitude characteristics.

Perform all the calculations required; plot out the results obtained on a graph of the amplitude characteristics with an appropriate scale (the y-axis corresponds to the phase shift).

QUESTIONS AND EXERCISES

1. What type of oscillations is called forced?

2. Derive the formulas of the amplitude and the phase difference between displacement and the driving force for linear forced oscillations excited by a force varying harmonically.

3. What is the essence of the resonance phenomena? What is its role in nature and technology?

4. What information about the oscillating system can be obtained from the resonance curve? How does the resonance curve change, when the damping coefficient is adjusted?

5. What are the conditions leading to necessity of accounting for the forces of inertia?