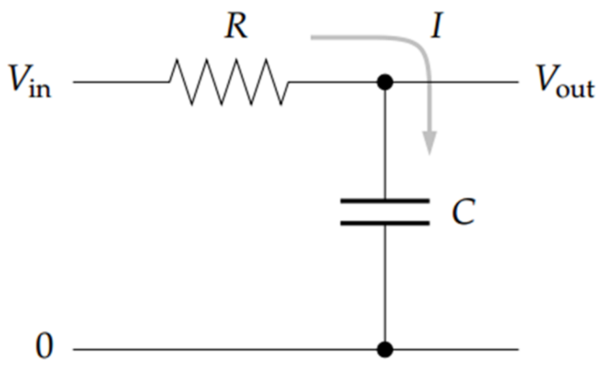
ODE & numerical methods

1. Motive force of a city car is 1.5 ton *(1 ton of force = 9800 N)*. Mass of the car is 10 ton. The friction force equals to $A \cdot v$, where v is velocity and A = 10 kg/s. Determine time during which the motion becomes uniform. Hint. Second order ODE can be transformed into a system of two ODEs of the first order.

2. Find out the speed which can be gained by a locomotive with the motive force of 25 ton *(1 ton of force = 9800 N)* during 60 s, if the total mass of the cars is 2000 tons? Consider the friction force noting that $F_{fr} = A \cdot v + B \cdot v^3$. Assume that $A = 10^4$ kg/s, and $B = 30$ kg·s/m^2. Find out the distance travelled by the train within first 3 min after leaving a station.

3. A simple electronic circuit with one resistor and one capacitor is depicted in the Fig.

This circuit acts as a low-pass filter: you send a signal in on the left and it comes out filtered on the right. Using Ohm's law and the capacitor law and assuming that the output load has very high impedance, so that a negligible amount of current flows through it, the equations governing this circuit are as follows:



$$IR = V_{in} - V_{out}\,,\quad Q = CV_{out}\,,\quad I = dQ/dt,$$

where I is the current that flows through R and into the capacitor, and Q is the charge on the capacitor.

Write a program to solve for Vout(t) when the input signal is a square-wave with frequency 1 and amplitude 1:

$$V_{in}(t) = \begin{cases} 1 & \text{if } \lfloor 2t \rfloor \text{ is even,} \\ -1 & \text{if } \lfloor 2t \rfloor \text{ is odd,} \end{cases}$$

where $\lfloor x \rfloor$ means x rounded off to the next lowest integer. Use the program to make plots of the output of the filter circuit from $t = 0$ to $t = 10$ when RC = 0.01, 0.1, and 1, with initial condition $V_{out}(0) = 0$. Make a decision about what value of h to use during the calculation. Try a variety of different steps. Based on the graphs produced by the program, describe what you see and explain what the circuit is doing.

4. **The driven pendulum problem**

   A pendulum can be driven by, for example, exerting a small oscillating force horizontally on the mass. Then assuming constants C and Ω, the equation of motion for the driven pendulum is (prove the formula):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta + C\cos\theta\sin\Omega t$$

   Write a program to solve this equation for θ as a function of time with ℓ = 10 cm, $C = 2$ s^{-2} and $\Omega = 5$ s^{-1} and make a plot of θ as a function of time from $t = 0$ to $t = 100$ s. Start the pendulum at rest with $\theta = 0$ and $d\theta/dt = 0$.

   As a next step, change the value of Ω, while keeping C the same, to find a value for which the pendulum resonates with the driving force and swings widely from side to side. Make a plot for this case as well.

5. **Trajectory with air resistance**

Consider a spherical cannonball shot from a cannon standing on the level ground. The air resistance on a moving sphere is a force in the opposite direction to the motion with magnitude

$$F = \tfrac{1}{2}\pi R^2 \rho C v^2,$$

where R is the sphere's radius, ρ is the density of air, v is the velocity, and C is the so-called *coefficient of drag* (a property of the shape of the moving object, in this case a sphere).

Starting from Newton's second law, $F = ma$, show that the equations of motion for the position (x, y) of the cannonball are:

$$\ddot{x} = -\frac{\pi R^2 \rho C}{2m}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}, \qquad \ddot{y} = -g - \frac{\pi R^2 \rho C}{2m}\dot{y}\sqrt{\dot{x}^2+\dot{y}^2},$$

where m is the mass of the cannonball, g is the acceleration due to gravity, and $\dot{x}$, $\ddot{x}$ are the first and second derivatives of x with respect to time.

Change these two second-order equations into four first-order equations, write a program that solves the equations for a cannonball of mass 1 kg and radius 8 cm, shot at 30º to the horizontal with initial velocity 100 $m\cdot s^{-1}$.The density of air is ρ = 1.22 $kg\cdot m^{-3}$ and the coefficient of drag for a sphere is C = 0.47. Make a plot of the trajectory of the cannonball (i.e., a graph of y as a function of x).

Use the program to estimate the total distance traveled (over the horizontal ground) by the cannonball above, and then experiment with the program to determine whether the cannonball travels further if it is heavier or lighter. Plot a series of trajectories for cannonballs of different masses. Make a plot of distance traveled as a function of the cannonball mass.

6. **Trajectory of a charged particle under influence of the electric and magnetic fields**

A particle ($m = 9\cdot10^{-31}$ kg, $q = +1.6\cdot10^{-19}$ C) enters magnetic field with the magnetic flux density of 0.1 T, perpendicular to the magnetic lines. Find the particle trajectory if its initial velocity was $0.2\cdot10^7$ m/s. Investigate dependence of the trajectory on the initial velocity ($v=0.2\cdot10^7$ m/s…$1.6\cdot10^7$ m/s). Add an electric field parallel to the magnetic one. Investigate how the trajectory would evolve changing the angle between the fields. Consider a general case when the initial velocity vector forms an angle α to the field.

7. **Trajectory of a charged particle under influence of a massive charged body**

Find trajectory of a charged particle (m = 1 g, $q = 1\cdot10^{-2}$ C) in the field set by a massive charged body with charge $Q = 5\cdot10^{-2}$ C. Initial conditions: distance between the charged particles 1 m, velocity of the small charged particle 10^{-1} m/s, angle between the radius-vector and velocity α = 30º. Modify the program assuming $q = -1\cdot10^{-2}$ C.

8. Soldering iron problem

Heat power of a soldering iron is 40 W, its mass m = 100 g, and its specific heat capacity is 400 J/kg/K. Heat is removed due to air convection and thermo-conductance: $dQ_1/dt = A(T-T_{amb})$, A = 0.05 W/K, ambient temperature T_{amb} = 20 ºC, and also by the Stephen-Boltzmann irradiation: $dQ_2/dt = S(T^4-T^4_{amb})$, with $S = 3.4\cdot10^{-10}$ W/K^4. Find how the iron temperature is changing with time.