Solving Ordinary Differential Equations Numerically

Ordinary Differential Equation (ODE)

$$y' = f(x,y)$$

$$y(x_0) = y_0$$ Initial (border condition)

To solve ODE, means to find *y(x)* – *aka Cauchy* problem

Approximate Methods:

- ***analytical***
- ***graphical***
- ***numerical***

***Picard Method** is an iterative method, and gives approximate solution in the analytical form*

$y' = f(x,y)$ → $\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)dx$

$y(x_0) = y_0$

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)dx$$

for $x = x_0$:

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(x,y)dx = y_0$$

let $y = y_0$:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x,y_0)dx$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x,y_1)dx$$

....

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x,y_{n-1})dx$$

aka
Iterative procedure
Method of successive iterations
Convergence method
Cut-and-try method

Picard Method ***is an iterative method, and gives approximate solution in the analytical form***

Example

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

$$y' = \frac{dy}{dx} = x^2 + 3y$$

$$y(x_0 = 0) = 2$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

y_0

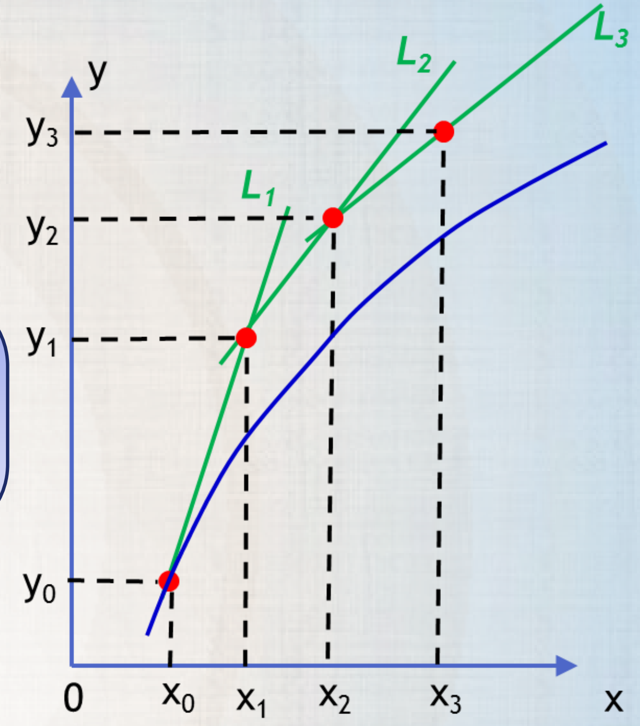
$$y_1(x) = \overbrace{2} + \int_0^x (x^2 + 3 \cdot \overbrace{2})dx = 2 + \frac{x^3}{3} + 6x$$

$$y_2(x) = \overbrace{2} + \int_0^x \left(x^2 + 3 \cdot \underbrace{\left(2 + \frac{x^3}{3} + 6x\right)}_{y_1}\right) dx = 2 + 6x + \frac{x^4}{4} + 9x^2 + \frac{x^3}{3}$$

$$y_3(x) = \overbrace{2} + \int_0^x \left(x^2 + 3 \cdot \underbrace{\left(2 + 6x + \frac{x^4}{4} + 9x^2 + \frac{x^3}{3}\right)}_{y_2}\right) dx = \cdots$$

....

Euler Method ***is an iterative method, and gives approximate solution in the graphical and table form***



$y' = f(x,y)$

$y(x_0) = y_0$

$$\begin{cases} \Delta y_k = h \cdot f(x_k, y_k) \\ y_{k+1} = y_k + \Delta y_k \end{cases}$$

Introduce step h : $x_i = x_0 + i \cdot h$

Tangent line L_1:

$$y = y_0 + f(x_0, y_0) \cdot (x - x_0)$$

for $x = x_1 : \ y_1 = y_0 + h \cdot f(x_0, y_0)$

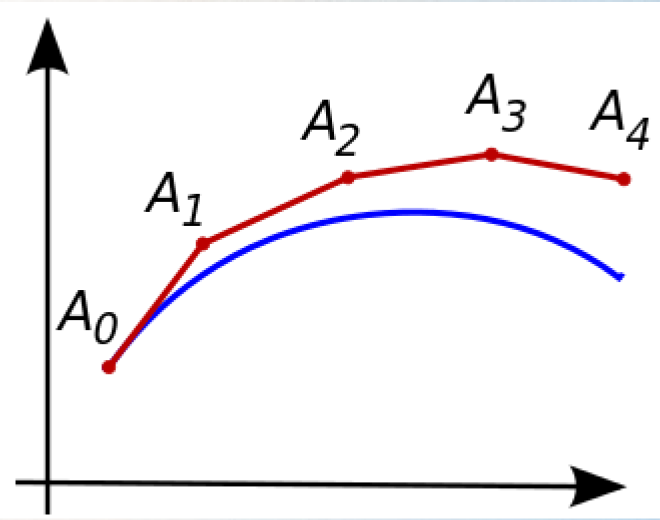
$$\Delta y_0 = h \cdot f(x_0, y_0)$$

Tangent line L_2:

$$y = y_1 + f(x_1, y_1) \cdot (x - x_1)$$

for $x = x_2 : \ y_2 = y_1 + h \cdot f(x_1, y_1)$

$$\Delta y_1 = h \cdot f(x_1, y_1)$$



Euler Method is an iterative method, and gives approximate solution in the graphical and table form

Example

$y' = \cos y + 3x$

$y(x_0 = 0) = 1.3$

$[\ 0\ ;\ 1\ ]$

⇓

choose step

$h = 0.2$

$$\begin{cases} \Delta y_k = h \cdot f(x_k, y_k) \\ y_{k+1} = y_k + \Delta y_k \end{cases}$$

k	x_k	y_k	$\Delta y_k = h \cdot [\cos(y_k)+3x_k$
0	0	1.30	0.05
1	0.2	1.35	0.16
2	0.4	1.52	0.25
3	0.6	1.77	0.32
4	0.8	2.09	0.38
5	1.0	2.47	

***Euler Method* is an iterative method, and gives approximate solution in the graphical and table form**

$$\begin{cases} \Delta y_k = h \cdot f(x_k, y_k) \\ y_{k+1} = y_k + \Delta y_k \end{cases}$$

Euler – Cauchy modification:

$$\begin{cases} y^*_{k+1} = y_k + h \cdot f(x_k, y_k) \\ y_{k+1} = y_k + h \cdot \dfrac{f(x_k, y_k) + f(x_{k+1}, y^*_{k+1})}{2} \end{cases}$$

***Runge - Kutta Method** is an iterative method, and gives approximate solution in the graphical and table form*

$y' = f(x,y)$

$y(x_0) = y_0$

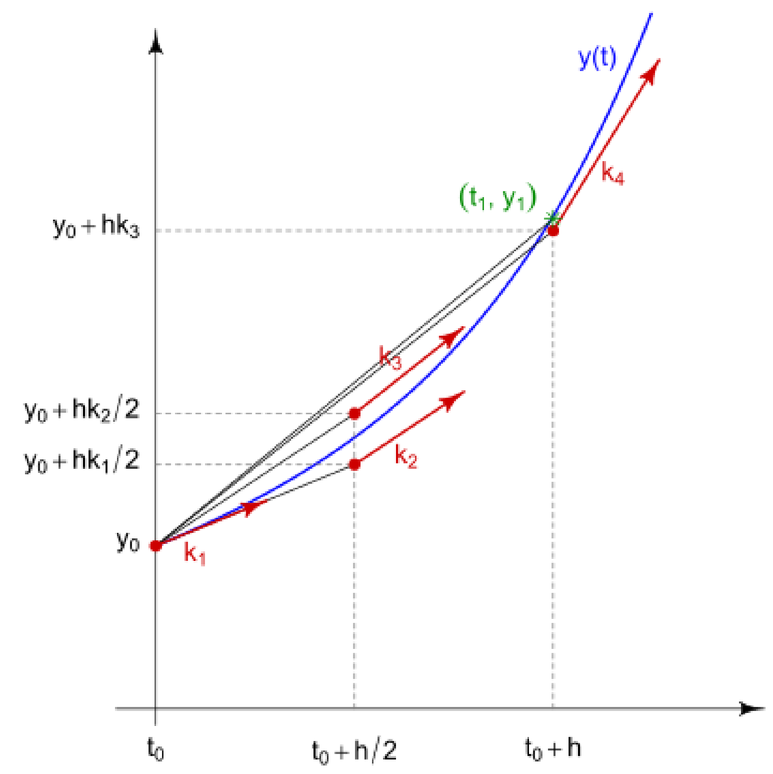
Introduce step h : $x_i = x_0 + i \cdot h$

$$y_{i+1} = y_i + \frac{1}{6} \cdot (k_1 + 2(k_2 + k_3) + k_4)$$

$$\begin{cases} k_1^{(i)} = h \cdot f(x_i, y_i) \\ k_2^{(i)} = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k_1^{(i)}}{2}\right) \\ k_3^{(i)} = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k_2^{(i)}}{2}\right) \\ k_4^{(i)} = h \cdot f\left(x_i + h, y_i + k_3^{(i)}\right) \end{cases}$$

***Runge - Kutta Method** is an iterative method, and gives approximate solution in the graphical and table form*

- k_1 is the slope at the beginning of the interval, using y (Euler's method);
- k_2 is the slope at the midpoint of the interval, using y and k_1 ;
- k_3 is again the slope at the midpoint, but now using y and k_2 ;
- k_4 is the slope at the end of the interval, using y and k_3 .



***Runge - Kutta Method** is an iterative method, and gives approximate solution in the graphical and table form*

Example 1

$y' = y(1 - x)$

$y(0) = 1$

[0;1]

Example 2

$y' = x + y\,(x)^{1/2}$

$y(0) = 0$

[0.0;0.6]

To do:

- ✓ ***Training tasks***
- ✓ ***Physics tasks (simple ones)***
- ✓ ***Physics tasks (advanced ones)***

Training tasks

Solving ODE Numerically

1. $y' = 3+2x-y$ $y(0)=2$, $x\in[0; 1]$, $h=0.2$	21. $y' = 3+2x+y$ $y(0)=2$ $x\in[0; 1]$ $h=0.2$
2. $y' = y-3x$ $y(1)=0$ $x\in[1; 2.2]$ $h=0.3$	22. $y' = 2y-x$ $y(1)=0$ $x\in[1; 2.2]$ $h=0.3$
3. $y' = 1-x+y$ $y(1,1)=0$ $x\in[1.1; 1.6]$ $h=0.1$	23. $y' = -x+y$ $y(1,1)=0$ $x\in[1.1; 1.6]$ $h=0.1$
4. $y' = y-7x$ $y(3)=3$ $x\in[3; 5]$ $h=0.5$	24. $y' = y-7x+2$ $y(3)=3$ $x\in[3; 5]$ $h=0.5$
5. $y' = 5-y+x$ $y(1)=1$ $x\in[1; 5]$ $h=1$	25. $y' = 5-y+x$ $y(1)=1$ $x\in[1; 5]$ $h=1$
6. $y' = y-2x+3$ $y(0)=4$ $x\in[0; 1]$ $h=0.2$	26. $y' = y-2x+3$ $y(0)=4$ $x\in[0; 1]$ $h=0.2$
7. $y' = 4-x+2y$ $y(0)=1$ $x\in[0; 1.2]$ $h=0.3$	27. $y' = 4-x+2y$ $y(0)=1$ $x\in[0; 1.2]$ $h=0.3$
8. $y' = -8+2x-y$ $y(1)=3$ $x\in[1; 3]$ $h=0.4$	28. $y' = -8+2x-y$ $y(1)=3$ $x\in[1; 3]$ $h=0.4$
9. $y' = 2y-3x$ $y(4)=0$ $x\in[4; 6]$ $h=0.5$	29. $y' = 2y-3x$ $y(4)=0$ $x\in[4; 6]$ $h=0.5$
10. $y' = x-2y$ $y(-1)=1$ $x\in[-1; 2]$ $h=0.6$	30. $y' = x^2-2y$ $y(-1)=1$ $x\in[-1; 2]$ $h=0.5$
11. $y' = 7-xy$ $y(-2)=0$ $x\in[-2; 0]$ $h=0.5$	31. $y' = 5-x-2y$ $y(1)=2$ $x\in[2; 4]$ $h=0.5$
12. $y' = 2x+y$ $y(2)=2$ $x\in[2; 3.5]$ $h=0.5$	32. $y' = y+3x-2$ $y(1)=2$ $x\in[1; 2]$ $h=0.2$
13. $y' = 5+x-y$ $y(2)=1$ $x\in[2; 4]$ $h=0.5$	33. $y' = y-2x$ $y(1)=2$ $x\in[1; 2.2]$ $h=0.3$
14. $y' = y+5x-1$ $y(0)=2$ $x\in[0; 3.2]$ $h=0.8$	34. $y' = 1-x+y$ $y(1,1)=1$ $x\in[1.1; 1.6]$ $h=0.1$
15. $y' = y-5x+1$ $y(0)=2$ $x\in[0; 3.2]$ $h=0.8$	35. $y' = y-7x$ $y(3)=1$ $x\in[3; 5]$ $h=0.5$
16. $y' = 1-x+y$ $y(0)=1$ $x\in[0; 2.5]$ $h=0.5$	36. $y' = x^2+0.1y^2$, $y(0)=0.7$, $x\in[0; 2]$ $h=0.1$
17. $y' = y-5x$ $y(-1)=1$ $x\in[-1; 1]$ $h=0.4$	37. $y' = 0.1x^2+2xy$, $y(0)=0.8$, $x\in[0; 2]$ $h=0.1$
18. $y' = x+2y$ $y(0)=-1$ $x\in[0; 2]$ $h=0.4$	38. $y' = x^2+y^2$, $y(0)=0.7$, $x\in[0; 2]$ $h=0.1$
19. $y' = x+y+2$ $y(1)=1$ $x\in[1; 3]$ $h=0.5$	39. $y' = xy+y^2$, $y(0)=0.6$, $x\in[0; 2]$ $h=0.1$
20. $y' = 3x+4y$ $y(2)=1$ $x\in[2; 5]$ $h=0.5$	40. $y' = 2x+0.1y^2$, $y(0)=0.2$, $x\in[0; 2]$ $h=0.1$

Physics tasks

1. A charge of 10^{-9} C was transposed to a conductor, however, the latter loses the charge due to some electrostatic leakage. The leakage rate is proportional to the charge on the conductor at the moment. During the first second $dq_{1s} = 10^{-10}$ C has leaked. What charge would be present on the conductor after Δt = 10 s?
2. A motorboat having speed of 40 km/h in a still water would slow down to 6 km/h once its engine is shut off during 20 s. Assuming friction force of the water proportional to boat's speed, find the boat speed after 2 min of inertial motion. Assume friction force to be $av+bv^3$. Guess reasonable values of a,b.
3. A point mass (m = 1 g) is moving rectilinearly, and the force is proportional to time t and inversely proportional to the velocity. At t=10 s, v = 0.5 m/s, F = $4\cdot10^{-5}$ N.
   Find $v(t=60\text{ s})$.
4. A point mass is moving rectilinearly under effect of a force proportional to t^3, where t is time (at $t_0=0$, $v=v_0$). There is also friction force proportional to vt. Assuming reasonable proportionality coefficients, find $v(t)$.
5. Find time needed for a body to cool down from 100 ºC to 25 ºC if the ambient temperature is 20 ºC, and 10 min were required for it to cool down from 100 ºC to 60 ºC.
6. Potential difference measured across a coil changed from 2 V down to 1 V within 1 s. Find the current after 10 s if I_0 = 17 A. Coil resistance is 0.12 Ω, inductance is 0.1 H.
   Assuming that $\varepsilon = \varepsilon_0\cdot\sin(\omega t)$, find current in 20 s from the beginning of the measurement.
7. A circuit consists of a power supply with e.m.f. ε, inductance L and resistance R connected in series. Assuming linearly increased ε, find current at t=10 s. L = 0.10 H, R = 50 Ω, I(t=0)=0 A. Note that potential difference across a conductor with resistance and inductance is $\Delta\varphi = L\frac{dI}{dt} + RI$.