Numerical Integration Methods

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x	x_0	x_1	x_2	…	x_n
$f(x)$	y_0	y_1	y_2	…	y_n

Numerical Integration –is used if:

- *the function is given in a table form;*
- *the integrand has rather a complicated form;*
- *numerical solution would be:*
  - *faster and simpler ,*
  - *more general…*

Numerical Integration –can be carried out by:

- *Substitution Method (integrand -> polynomial function)*
- *Method of undetermined coefficients*
- *Monte Carlo Methods*

Numerical Integration carried out by Substitution Method (integrand -> polynomial function) using the Lagrange Interpolation :

- **Of the 1st order -> trapezoidal rule**
- **Of the 2nd order -> Simpson's rule**

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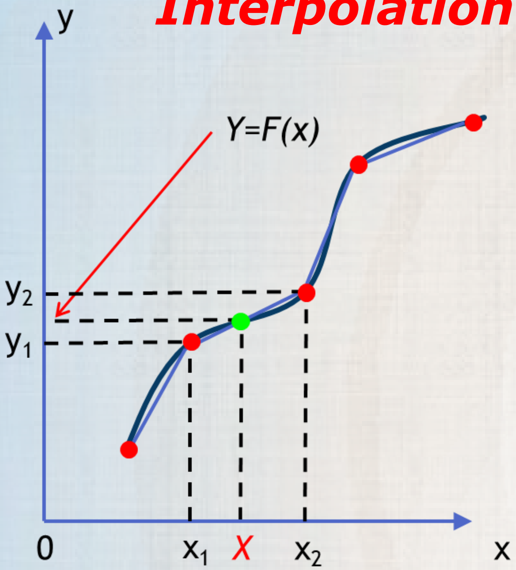
Interpolation: Lagrange formula



x	x_0	x_1	x_2	…	x_n
$f(x)$	y_0	y_1	y_2	…	y_n

$$F(x) \rightarrow P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a^{n-1}x + a_n$$

$$L_n(x) = \ell_0(x) + \ell_1(x) + \ell_2(x) + \ldots + \ell_n(x)$$

where $\ell_i(x_k) = \begin{cases} y_i, & i=k \\ 0, & i \neq k \end{cases}$

$$\ell_i(x) = C_i \cdot (x-x_0)(x-x_1)(x-x_2) \cdot\cdot\cdot (x-x_{i-1})(x-x_{i+1}) \cdot\cdot\cdot (x-x_n)$$

$$C_i = \frac{y_i}{(x_i-x_0)(x_i-x_1)(x_i-x_2) \cdot\cdot\cdot (x_i-x_{i-1})(x_i-x_{i+1}) \cdot\cdot\cdot (x_i-x_n)}$$

$$L_n = \sum_{i=0}^{n} y_i \frac{(x-x_0)(x-x_1)(x-x_2) \cdot\cdot\cdot (x-x_{i-1})(x-x_{i+1}) \cdot\cdot\cdot (x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2) \cdot\cdot\cdot (x_i-x_{i-1})(x_i-x_{i+1}) \cdot\cdot\cdot (x_i-x_n)}$$

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Integration: Trapezoidal rule

x	x_0	x_1	x_2	…	x_n
f(x)	y_0	y_1	y_2	…	y_n

$$\int_{a=x_0}^{b=x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$h = \frac{x_n - x_0}{n}$$

$$\int_{x_0}^{x_1} f(x)dx \approx$$

$$\int_{x_0}^{x_1} L_1 f(x)dx = \int_{x_0}^{x_1} \left[\frac{x-x_1}{x_0-x_1} \cdot y_0 + \frac{x-x_0}{x_1-x_0} \cdot y_1\right] dx = \frac{h}{2} \cdot (y_0 + y_1)$$

$$\mathrm{L}_n = \sum_{i=0}^{n} y_i \frac{(x-x_0)(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_{i-1})(x-x_{i+1})\cdot\ldots\cdot(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\cdot\ldots\cdot(x_i-x_{i-1})(x_i-x_{i+1})\cdot\ldots\cdot(x_i-x_n)}$$

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Integration: Trapezoidal rule

x	x_0	x_1	x_2	…	x_n
f(x)	y_0	y_1	y_2	…	y_n

$$h = \frac{x_n - x_0}{n}$$

$$\int_{a=x_0}^{b=x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}\cdot(y_0 + y_1)$$

$$\int_{x_1}^{x_2} f(x)dx \approx \frac{h}{2}\cdot(y_1 + y_2)$$

$$\int_{x_2}^{x_3} f(x)dx \approx \frac{h}{2}\cdot(y_2 + y_3)$$

…

$$\int_{x_{n-1}}^{x_n} f(x)dx \approx \frac{h}{2}\cdot(y_{n-1} + y_n)$$

$$\int_{a=x_0}^{b=x_n} f(x)dx \approx \frac{h}{2}\cdot(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i\right] = \frac{h}{2}y_0 + \frac{h}{2}y_n + h\cdot\sum_{i=1}^{n-1} y_i$$

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Interpolation: Lagrange formula

$$L_n = \sum_{i=0}^{n} y_i \frac{(x-x_0)(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_{i-1})(x-x_{i+1})\cdot\ldots\cdot(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\cdot\ldots\cdot(x_i-x_{i-1})(x_i-x_{i+1})\cdot\ldots\cdot(x_i-x_n)}$$

Numerical term C_i

Functional term

coefficients

$$l_0 = y_0 \cdot \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{y_0}{(x_0-x_1)(x_0-x_2)} \cdot \frac{x^2 + x\cdot(-x_1-x_2) + x_1\cdot x_2}{1}$$

$$l_1 = y_1 \cdot \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{y_1}{(x_1-x_0)(x_1-x_2)} \cdot \frac{x^2 + x\cdot(-x_0-x_2) + x_0\cdot x_2}{1}$$

$$l_2 = y_2 \cdot \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{y_1}{(x_2-x_0)(x_2-x_1)} \cdot \frac{x^2 + x\cdot(-x_0-x_1) + x_0\cdot x_1}{1}$$

$$\mathbf{L}_n = \boldsymbol{l_0 + l_1 + l_2}$$

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Integration: Simpson's rule

x	x_0	x_1	x_2	…	x_{2n}
f(x)	y_0	y_1	y_2	…	y_{2n}

Use even number of intervals

$$\int_{a=x_0}^{b=x_{2n}} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx$$

$$h = \frac{x_{2n} - x_0}{2n}$$

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} L_2 f(x)dx =$$

$$= \int_{x_0}^{x_2} \left[\frac{(x-x_1)\cdot(x-x_2)}{(x_0-x_1)\cdot(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)\cdot(x-x_2)}{(x_1-x_0)\cdot(x_1-x_2)} \cdot y_1 + \frac{(x-x_0)\cdot(x-x_1)}{(x_2-x_0)\cdot(x_2-x_1)} \cdot y_2\right] dx =$$

$$= \frac{h}{3} \cdot (y_0 + 4y_1 + y_2)$$

$$\mathrm{L}_n = \sum_{i=0}^{n} y_i \frac{(x-x_0)(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_{i-1})(x-x_{i+1})\cdot\ldots\cdot(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\cdot\ldots\cdot(x_i-x_{i-1})(x_i-x_{i+1})\cdot\ldots\cdot(x_i-x_n)}$$

Integration: Simpson's rule

x	x_0	x_1	x_2	...	x_{2n}
$f(x)$	y_0	y_1	y_2	...	y_{2n}

Use even number of intervals

$$\int_{a=x_0}^{b=x_{2n}} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx$$

$$h = \frac{x_{2n} - x_0}{2n}$$

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3} \cdot (y_0 + 4y_1 + y_2)$$

$$\int_{x_2}^{x_4} f(x)dx \approx \frac{h}{3} \cdot (y_2 + 4y_3 + y_4)$$

...

$$\int_{x_{2n-2}}^{x_{2n}} f(x)dx \approx \frac{h}{3} \cdot (y_{2n-2} + 4y_{2n-1} + y_{2n})$$

$$\int_{a=x_0}^{b=x_{2n}} f(x)dx \approx \frac{h}{3} \cdot (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 \ldots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}) =$$

$$= \frac{h}{3}\left[y_0 + y_{2n} + (K) \cdot \sum_{i=1}^{2n-1} y_i\right]$$

K can be constructed using math operators

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To do:

- ✓ ***Training tasks***
- ✓ ***Velocities problem***
- ✓ ***Physics tasks***

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Training tasks

1. Mathematical exercises. Compute the integrals using trapezoidal and Simpson rules. Use the specified number of intervals *n, 2n, 5n, 10n, 20n*.

1.	$\int_{-2}^{4}(2x^2-\sqrt{x+2})dx$	$n=6$	2.	$\int_{-3}^{0}(5x^2+x+1)dx$	$n=6$
3.	$\int_{0}^{3}(3x^2-\sqrt{x})dx$	$n=6$	4.	$\int_{1}^{4}(x^3-\sqrt{x})dx$	$n=6$
5.	$\int_{1}^{4}(7+x-2x^2)dx$	$n=6$	6.	$\int_{0}^{3}(7x^2-3\sqrt{x})dx$	$n=6$
7.	$\int_{2}^{5}(2x^2-2-\sqrt{x})dx$	$n=6$	8.	$\int_{0}^{3}(5x^2+\sqrt{x})dx$	$n=6$
9.	$\int_{-2}^{2}(x^3+1)dx$	$n=8$	10.	$\int_{0}^{4}(2x^2+1-\sqrt{x})dx$	$n=8$
11.	$\int_{-2}^{2}(x^2+\sqrt{x+2}-1)dx$	$n=8$	12.	$\int_{0}^{2}(x^2+2+\sqrt{x})dx$	$n=8$
13.	$\int_{1}^{3}(3x^2-x-1)dx$	$n=8$	14.	$\int_{-1}^{3}(x^3+2)dx$	$n=8$
15.	$\int_{-2}^{2}(2x^2+1-\sqrt{x+4})dx$	$n=8$	16.	$\int_{1}^{4}(2x^2-1{,}5\sqrt{x})dx$	$n=6$
17.	$\int_{1}^{4}(7\sqrt{x}+2x^2)dx$	$n=6$	18.	$\int_{0}^{3}(7x^2+3\sqrt{x})dx$	$n=6$
19.	$\int_{2}^{5}(2x^2-2+\sqrt{x})dx$	$n=6$	20.	$\int_{0}^{3}(5x^2-1+\sqrt{x})dx$	$n=6$
21.	$\int_{3}^{6}(x^2+4+\sqrt{x})dx$	$n=6$	22.	$\int_{2}^{6}(x^3+3)dx$	$n=8$
23.	$\int_{0}^{3}(2x^2-1+\sqrt{x})dx$	$n=6$	24.	$\int_{-2}^{2}(3x^2+2\sqrt{x+2})dx$	$n=8$

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Velocities problem

2. In the file “velocities.txt”

(http://www-personal.umich.edu/~mejn/cp/data/velocities.txt),

the first column represents time *t* in seconds and the second - the *x*-velocity in meters per second of a particle, measured once every second from time $t = 0$ to $t = 100$.

Read in the data and, using numerical integration, calculate the approximate distance traveled by the particle in the *x* direction as a function of time.

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Physics tasks

1. Find area of a flat figure D defined by the curvatures given below and noting that

$$S_D = \iint_D dx\,dy$$

1) $\begin{cases} y = x^2 \\ y = \frac{3-x}{2} \\ y = 0 \end{cases}$

2) $\begin{cases} x = 2 \\ y = x \\ xy = 1 \end{cases}$

3) $\begin{cases} y = x \\ y = 5x \\ y = 1 \end{cases}$

4) $\begin{cases} x = 4 - y^2 \\ x + 2y - 4 = 0 \end{cases}$

5) $\begin{cases} x = y^2 - 2y \\ x + y = 0 \end{cases}$

6) $\begin{cases} x = 4y - y^2 \\ x + y = 6 \\ y = 0 \end{cases}$

7) $\begin{cases} x + y = 2 \\ y = x \\ y = 0 \end{cases}$

8) $\begin{cases} x = 0 \\ y = \pi \\ y = x \end{cases}$

9) $\begin{cases} y = \frac{5}{9} \cdot x^2 \\ x = \frac{3}{25} \cdot y^2 \end{cases}$

10) $\begin{cases} x + y - 5 = 0 \\ xy = 4 \end{cases}$

11) $\begin{cases} y^2 = 10x + 25 \\ y^2 + 6x = 9 \end{cases}$

12) $\begin{cases} y = \frac{\sqrt{x}}{2} \\ y = \frac{1}{2x} \\ x = 16 \end{cases}$

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Physics tasks

2. Find moment of inertia (with respect to the Cartesian axes) of a flat plate defined by the given curvatures. Note that

$$\begin{cases} J_X = \iint\limits_D y^2 \rho(x,y)\,dx\,dy \\ J_Y = \iint\limits_D x^2 \rho(x,y)\,dx\,dy \end{cases}$$

1) $\begin{cases} x = 2 \\ y = 0 \\ x = 2y^2 \ (y \geq 0) \\ \rho(x,y) = \frac{7x^2}{2} + 6y \end{cases}$

3) $\begin{cases} y = 2x - x^2 \\ y = x \\ \rho(x,y) = x \end{cases}$

5) $\begin{cases} y = 2x^3 \\ y^2 = 2x \\ \rho(x,y) = y^2 \end{cases}$

7) $\begin{cases} \frac{x^2}{4} + y^2 = 1 \\ \rho(x,y) = 4y^2 \end{cases}$

2) $\begin{cases} y = x^2 \\ x = y^2 \\ \rho(x,y) = x \end{cases}$

4) $\begin{cases} x = 0 \\ x = 2 \\ y = 0 \\ y = 3 \\ \rho(x,y) = x^3 y^2 \end{cases}$

6) $\begin{cases} x = y^2 \\ y = x^2 \\ \rho(x,y) = y \end{cases}$

8) $\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \\ \rho(x,y) = \frac{x+y}{x^2+y^2} \end{cases}$

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Physics tasks

3. Find coordinates of the centre of the mass of a flat plate ($x_c = M_Y/m$, $y_c = M_X/m$) using

$$\begin{cases} M_X = \iint\limits_D y\rho(x,y)\,dx\,dy \\ M_Y = \iint\limits_D x\rho(x,y)\,dx\,dy \\ m = \iint\limits_D \rho(x,y)\,dx\,dy \end{cases}$$

1) $\begin{cases} y = \sqrt{x} \\ y = 2 - x \\ x = 0 \\ \rho(x,y) = y \end{cases}$

2) $\begin{cases} y = 4 - x^2 \\ y = 0 \\ \rho(x,y) = x \end{cases}$

3) $\begin{cases} y = 2\sqrt{x} \\ x + y = 3 \\ y = 0 \\ \rho(x,y) = 1/x \end{cases}$

4) $\begin{cases} y = x^3 \\ y = -\sqrt{x} \\ x = 2 \\ \rho(x,y) = x^3 \end{cases}$

5) $\begin{cases} y = x^2 \\ y = 2 - x \\ \rho(x,y) = y^2 \end{cases}$

6) $\begin{cases} y = 2x - x^2 \\ y = 2 - x^2 \\ x = 0 \\ \rho(x,y) = y \end{cases}$

7) $\begin{cases} x^2 + y^2 = 2x \\ \rho(x,y) = 4y^2 \end{cases}$

8) $\begin{cases} x^2 + y^2 = 2x \\ x^2 + y^2 = 6x \\ \rho(x,y) = \frac{x^2+y^2}{x+y} \end{cases}$

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Physics tasks

4. Find the moment of inertia of a homogeneous ball, with mass m.

5. Find the moment of inertia of an inhomogeneous ball, with density ρ:

   1) $\rho = \rho_0 \cdot e^{-r}$
   2) $\rho = \rho_0 \cdot e^{-\frac{r^2}{2}+2r}$

6. Find the charge flown through a circuit during T = 1, 10, 100 min if

   1) $I = I_0 \cdot e^{-\alpha \cdot \frac{t}{T}}$
   2) $\begin{cases} U = U_0 \cdot e^{-\alpha \cdot \frac{t}{T}} \\ R = R_0 \cdot \frac{\sqrt{T}}{\sqrt{t}} \end{cases}$

   Vary parameters I_0, U_0, R_0, α within a meaningful range.

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